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On the Order of Magnitude of Functions at Infinity

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1. Our purpose is to relate, in quite a general setting, the order of magnitude of real functions $f(x)$ as $x \to \infty$ to their degree of approximation by piecewise polynomials interpolating them on some simple denumerable sets of points.

2. Let f be a real function on $[0, \infty)$, let k be a positive integer, and let h be a real function satisfying $h(0) = 0$, $h'(x) > 0$ and nonincreasing in $[0, \infty)$, and $\lim_{x \to \infty} h(x) = \infty$. We denote by $P_k(f, x; h) \equiv P_k(f)$ the function with domain $[0, \infty)$ which in each

$$I_n = [h(n-1), h(n)), \qquad n = 1, 2, 3, \ldots, \tag{1}$$

coincides with the polynomial of degree $\leqslant k$ interpolating f at the $k+1$ equally spaced points

$$x_j^{(n)} = h(n-1) + (d_n/k)j, \qquad j = 0, 1, \ldots, k, \tag{2}$$

where $d_n = h(n) - h(n-1)$ is the length of I_n. In particular, $P_1(f)$ is a polygonal function, interpolating f at $h(n)$, $n = 0, 1, 2, \ldots$. In the following theorem we relate the order of magnitude of $f(x)$ as $x \to \infty$ to that of our "degree of approximation"

$$\langle f \rangle_{k,y} \equiv \sup_{x > y} |f(x) - P_k(f, x; h)|$$

as $y \to \infty$.

Later we show that, in our theorem (in one direction), $P_k(f)$ can be replaced by *any* piecewise polynomial of degree $\leqslant k$ whose knots are $h(n)$, $n = 0, 1, 2, \ldots$, not necessarily one arising from interpolation.

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3. THEOREM. *Let $f^{(k+1)}$ exist and be $\geqslant 0$ and nondecreasing (or $\leqslant 0$ and nonincreasing) in $[0, \infty)$. Let g be a real function satisfying*

$$g(0) > 0, \qquad g'(x) > 0 \qquad \text{on} \quad [0, \infty). \tag{3}$$

$$x^k = O(g(x)) \qquad \text{as} \quad x \to \infty. \tag{4}$$

g'/g is nondecreasing in $[0, \infty)$ and absolutely continuous in each $[0, x]$, $\quad 0 < x < \infty$. (5)

There is a constant A such that, for $n = 1, 2, \ldots,$

$$h'(n-1) \leqslant Ah'(n). \tag{6}$$

There are constants $B(\geqslant 0), C, D$ such that, for every $x \geqslant B$, there is a $t_x > x$ satisfying

$$g'(t_x)/g(t_x) \leqslant Cg'(x)/g(x), \qquad g(t_x)/(t_x - x) \leqslant Dg'(x). \tag{7}$$

There is a constant E such that $\phi(y) \leqslant E\phi(x)$ whenever $0 \leqslant x < y$. (8)

Here

$$\phi(x) = [h'(h^{-1}(x))\, g'(x)]^{k+1}\, g^{-k}(x) \qquad \text{on} \quad [0, \infty). \tag{9}$$

Then

$$f(x) = O(g(x)) \quad \textit{as} \quad x \to \infty \quad \textit{iff} \quad \langle f \rangle_{k,y} = O(\phi(y)) \quad \textit{as} \quad y \to \infty. \tag{10}$$

4. EXAMPLES. I. Let $h(x) \equiv \log(1+x)$, $0 < \alpha \leqslant k+1$, and $g(x) \equiv e^{\alpha x}$. In (7) and (8) one can take $B = 0$, $C = 1$, $D = e^{\alpha}/\alpha$, $t_x \equiv 1 + x$, and $E = 1$. Then (10) gives

$$f(x) = O(e^{\alpha x}) \quad \text{as} \quad x \to \infty \quad \text{iff} \quad \langle f \rangle_{k,y} = O(e^{(\alpha - k - 1) y}) \quad \text{as} \quad y \to \infty.$$

II. Let $h(x) \equiv \log\log(e + x)$, $g(x) \equiv e^{e^x}$. In (7) and (8) one can take $t_x \equiv x + e^{-x}$, $C = e$, D any number $> e$, B a sufficiently large number, and $E = 1$. Here (10) reads

$$f(x) = O(e^{e^x}) \quad \text{as} \quad x \to \infty \quad \text{iff} \quad \langle f \rangle_{k,y} = O(e^{-ke^y}) \quad \text{as} \quad y \to \infty.$$

5. *Proof of the Theorem.* Assume that $f^{(k+1)}$ is $\geqslant 0$ and nondecreasing in $[0, \infty)$ (otherwise, consider $-f$). Let

$$F(x) \equiv f(x) - \sum_{j=0}^{k} \frac{f^{(j)}(0)}{j!} x^j$$

so that $F(0) = F'(0) = \cdots = F^{(k)}(0) = 0$ and $F^{(k+1)}(x) \equiv f^{(k+1)}(x)$. Also $F - P_k(F) = f - P_k(f)$. Furthermore, by (4), $f(x) = O(g(x))$ as $x \to \infty$ iff $F(x) = O(g(x))$ as $x \to \infty$. Thus we may assume without loss of generality that

$$f^{(j)}(0) = 0,\ j = 0, 1, \dots, k, \text{ and hence } f^{(j)}(x) \text{ is } \geqslant 0 \text{ and nondecreasing in } [0, \infty) \text{ for } j = 0, 1, \dots, k+1. \tag{11}$$

Suppose now that M is a number such that

$$\langle f \rangle_{k,y} \leqslant M\phi(y) \qquad \text{for all} \quad y \geqslant \text{some } y_0 \geqslant 0. \tag{12}$$

Let $x \geqslant y_0$. Define the integer $n (\geqslant 1)$ and the number $\tilde{x}$ by

$$h(n-1) \leqslant x < h(n),$$
$$\tilde{x} = h(n) + (2k)^{-1}(h(n+1) - h(n)).$$

By the remainder theorem for Lagrange interpolation [1, p. 56] we have, using the notation (2), for some $\xi \in (h(n), h(n+1))$,

$$|f(\tilde{x}) - P_k(f, \tilde{x}; h)| = \frac{f^{(k+1)}(\xi)}{(k+1)!} \prod_{j=0}^{k} |\tilde{x} - x_j^{(n+1)}|$$
$$= \frac{f^{(k+1)}(\xi)}{(k+1)!} [h(n+1) - h(n)]^{k+1} \frac{1 \cdot 3 \cdots (2k-1)}{(2k)^{k+1}}.$$

By (6), $h(n+1) - h(n) = \int_n^{n+1} h'(x)\, dx \geqslant h'(n+1) \geqslant A^{-2} h'(n-1)$ so that

$$f^{(k+1)}(x) \leqslant f^{(k+1)}(\xi) \leqslant M_k |f(\tilde{x}) - P_k(f, \tilde{x}; h)|\, h'(n-1)^{-k-1}$$
$$\leqslant M_k \langle f \rangle_{k,x} h'(n-1)^{-k-1}$$

where $M_k = (k+1)!(2k)^{k+1} A^{2(k+1)}(1 \cdot 3 \cdots (2k-1))^{-1}$. By (12) and (9), $f^{(k+1)}(x) \leqslant MM_k\phi(x)\, h'(n-1)^{-k-1} \leqslant MM_k g'(x)^{k+1} g^{-k}(x)$. Thus, by (5), for some constant μ_k, $f^{(k+1)}(x) \leqslant \mu_k g'(x)^{k+1} g^{-k}(x)$ throughout $[0, \infty)$. Furthermore, for $j = 1, 2, \dots, k+2$, we have on $[0, \infty)$,

$$f^{(k+2-j)}(x) \leqslant \mu_k g'(x)^{k+2-j} g^{j-k-1}(x). \tag{13}$$

This was just shown to hold for $j=1$. Suppose it holds for some j, $1 \leqslant j \leqslant k+1$. Then, by (13), on $[0, \infty)$,

$$\begin{aligned} f^{(k+1-j)}(x) &= \int_0^x f^{(k+2-j)}(t)\,dt \leqslant \mu_k \int_0^x g'(t)[g'(t)/g(t)]^{k+1-j}\,dt \\ &= \mu_k\, g(t)[g'(t)/g(t)]^{k+1-j}\Big|_0^x \\ &\quad - \mu_k \int_0^x g(t)\frac{d}{dt}[\{g'(t)/g(t)\}^{k+1-j}]\,dt \\ &\leqslant \mu_k\, g(x)[g'(x)/g(x)]^{k+1-j}. \end{aligned}$$

Taking, in (13), $j=k+2$, we have by (11),

$$0 \leqslant f(x) \leqslant \mu_k\, g(x) \qquad \text{throughout} \quad [0, \infty),$$

as claimed in (10).

For the converse suppose that, for some constant J,

$$0 \leqslant f(x) \leqslant Jg(x) \qquad \text{throughout} \quad [0, \infty).$$

For $j=0, 1, \ldots, k+1$ and with B, C, D of (7),

$$f^{(j)}(x) \leqslant JC^{(j-1)j/2} D^j g'(x)^j g^{1-j}(x) \qquad \text{throughout} \quad [B, \infty). \tag{14}$$

This is true for $j=0$ and assuming its truth for some j, $0 \leqslant j \leqslant k$, we have by (11) and (7), for every $x \in [B, \infty)$ and a suitable $\theta \in (x, t_x)$,

$$\begin{aligned} JC^{(j-1)j/2} D^j g'(t_x)^j g^{1-j}(t_x) &\geqslant f^{(j)}(t_x) - f^{(j)}(x) = (t_x - x) f^{(j+1)}(\theta) \\ &\geqslant (t_x - x) f^{(j+1)}(x), \\ f^{(j+1)}(x) &\leqslant JC^{(j-1)j/2} D^j [g'(t_x)/g(t_x)]^j g(t_x)/(t_x - x) \\ &\leqslant JC^{j(j+1)/2} D^{j+1} g'(x)^{j+1} g^{-j}(x). \end{aligned}$$

With $J_k = JC^{k(k+1)/2} D^{k+1}$, (14) yields

$$f^{(k+1)}(x) \leqslant J_k [g'(x)/g(x)]^{k+1} g(x) \qquad \text{throughout} \quad [B, \infty)$$

and hence, by (3) and (5), for a suitable constant L,

$$f^{(k+1)}(x) \leqslant L[g'(x)/g(x)]^{k+1} g(x) \qquad \text{throughout} \quad [0, \infty). \tag{15}$$

Let $0 \leqslant \gamma \leqslant x$. For a proper $n \geqslant 1$, $h(n-1) \leqslant x < h(n)$. Using again (2) and the above remainder theorem, we have, for some $\eta \in (h(n-1), h(n))$,

$$|f(x) - P_k(f, x; h)| = [f^{(k+1)}(\eta)/(k+1)!] \prod_{j=0}^{k} |x - x_j^{(n)}|. \tag{16}$$

For $j = 0, 1, \ldots, k$, $|x - x_j^{(n)}| \leqslant h(n) - h(n-1) = \int_{n-1}^{n} h'(t)\,dt \leqslant Ah'(n)$ (by (6)). Setting $M = LA^{k+1}/(k+1)!$ we obtain from (16), (15), (5), (3), and (8),

$$\begin{aligned} |f(x) - P_k(f, x; h)| &\leqslant M \left[\frac{g'(\eta)}{g(\eta)}\right]^{k+1} g(\eta)\, h'(n)^{k+1} \\ &\leqslant M \left[\frac{g'(h(n))}{g(h(n))}\right]^{k+1} g(h(n))\, h'(n)^{k+1} \\ &= M\phi(h(n)) \leqslant ME\phi(\gamma). \end{aligned}$$

Hence $\langle f \rangle_{k,\gamma} = O(\phi(\gamma))$ as $\gamma \to \infty$.

6. COROLLARY. *Assume the hypotheses of the Theorem. A necessary and sufficient condition for $f(x)$ to be $O(g(x))$ as $x \to \infty$ is the existence of a real function $Q(x)$ with domain $[0, \infty)$, continuous there, which in each I_n of (1) coincides with some polynomial of degree $\leqslant k$ and such that*

$$\sup_{x \geqslant \gamma} |f(x) - Q(x)| = O(\phi(\gamma))$$

as $\gamma \to \infty$.

Proof. Only sufficiency needs proof. Let μ be a number such that

$$\sup_{x \geqslant \gamma} |f(x) - Q(x)| \leqslant \mu\phi(\gamma) \qquad \text{for all} \quad \gamma \geqslant 0. \tag{17}$$

Let $t \geqslant \gamma \geqslant 0$ and set

$$R(x) \equiv P_k(f, x; h) - Q(x).$$

Then $t \in I_n$ for some $n \geqslant 1$ and, using (2),

$$R(t) = \sum_{j=0}^{k} R(x_j^{(n)}) \prod_{\substack{s=0 \\ s \neq j}}^{k} (t - x_s^{(n)})/(x_j^{(n)} - x_s^{(n)}). \tag{18}$$

Let $0 \leqslant j \leqslant k$. If $x_j^{(n)} < \gamma$, then $h(n-1) < \gamma < h(n)$ and by (6), (5), and (3),

$$h'(n-1) \leqslant Ah'(n) \leqslant Ah'(h^{-1}(\gamma))$$

and

$$[g'(h(n-1))/g(h(n-1))]^{k+1} g(h(n-1)) < [g'(\gamma)/g(\gamma)]^{k+1} g(\gamma)$$

so that, by (17),

$$\begin{aligned} |f(x_j^{(n)}) - Q(x_j^{(n)})| &\leqslant \mu\phi(h(n-1)) \\ &= \mu[h'(n-1)\, g'(h(n-1))]^{k+1} g^{-k}(h(n-1)) \\ &\leqslant \mu A^{k+1}[h'(h^{-1}(\gamma))\, g'(\gamma)]^{k+1} g^{-k}(\gamma) = \mu A^{k+1}\, \phi(\gamma). \end{aligned}$$

If $x_j^{(n)} \geqslant \gamma$, then by (17), $|f(x_j^{(n)}) - Q(x_j^{(n)})| \leqslant \mu\phi(\gamma) \leqslant \mu A^{k+1}\phi(\gamma)$ as $A \geqslant 1$.

By (18),

$$|R(t)| \leqslant (k+1)\, \mu A^{k+1}\phi(\gamma)\, k^k$$

and hence by (17),

$$|f(t) - P_k(f, t; h)| \leqslant [1 + (k+1)\, A^{k+1}\, k^k]\, \mu\phi(\gamma).$$

Thus $\langle f \rangle_{k,\gamma} = O(\phi(\gamma))$ as $\gamma \to \infty$ and hence, by (10), $f(x) = O(g(x))$ as $x \to \infty$.

References

1. P. J. Davis, "Interpolation and Approximation," Blaisdell, New York, 1963; Dover, New York, 1975.